A-Level Further Maths Preparation and Transition Work

At The Thomas Adams School you will be studying your Further Maths A-Level alongside your standard Maths A-Level. Students do often find the transition from GCSE to A-Level maths to be challenging, and to make the transition smoother it is essential that you work through this transition work. This means that we can spend less time revisiting and revising GCSE topics and spend more time learning and investigating the new, more challenging content.

In Further Maths we go deeper into mathematics and study topics not seen in the standard Maths A-Level. One such topic is the topic of complex numbers which is the main focus of this transition work. The complex numbers work starts on the next page. You need to read through the examples, copy them down and then complete the exercise questions. There is then a smaller section on sequences that you should also complete.

If you get stuck, do not panic. This is a lot to ask of you, however, you are up to the challenge! Using the Internet and researching topics online is not cheating, it is a tool that you should use to help you gain knowledge and understanding. If the examples here are not quite enough for you to be able to complete the exercise questions, then Google the topic or watch YouTube videos. This shows initiative, resilience and dedication. A key ingredient for success in A-Level Maths and Further Maths is dedication and the willingness to put in work outside of lessons. Show me your commitment and any problems that you really cannot solve we will look over when you arrive in September.

This work (and the standard Maths A-Level transition work) must be completed before you arrive in September. I will be asking you to hand in your solutions. I look forward to meeting you all in September where you are about to embark on one of the most interesting A-Level subjects available.

1. Complex Numbers

You may have noticed that there is no solution to equations like $x^2 = -4$. Naively we may try letting $x = -2$ but $(-2)^2 = 4$ not -4. The difficulty here is that there is no number that can be squared to give a negative answer, or equivalently you 'cannot square root a negative number'.

However, there is a solution, we just need to create a number which is the square root of a negative number. Let us define a new number i so that $i^2 = -1$ or equivalently $i = \sqrt{-1}$. This may seem like a ridiculous thing to do so let me take a moment to explain why this isn't as strange as it may first appear.

Think back to when you first started learning Maths. You probably started with positive whole numbers $1, 2, 3, 4, \ldots$ these are called the natural numbers or $\mathbb{N}$ in symbol form. At some point your teacher would have asked you to do a calculation like $3-5$ and you may well have said that it was impossible - you cannot take away a larger number from a smaller number. Of course you now know you can, you just end up with a new number, we call them negative numbers.

So now we have whole numbers $-4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$ both positive and negative called integers or $\mathbb{Z}$. Next someone will have asked you to do a division like $\frac{5}{3}$ and you may have thought that it was impossible since 3 isn't a factor of 5, so 3 doesn't divide 5. Again, we now know that this is also ok, you just don't get a integer as the answer, instead you get a fraction.

Now you have fractions and I will struggle to list them all but here are a few of them:

$$\begin{matrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \ldots \\ \frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} \\ \frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} \end{matrix}$$

These fractions are also called rational numbers or quotients and have symbol $\mathbb{Q}$.

You may also know about numbers like π and $\sqrt{2}$ which cannot be written as a fraction so are not rational numbers, they are called irrational numbers and have infinite decimal expansions with no recurring pattern. If we put rationals and irrationals together we get a set of numbers called real numbers or $\mathbb{R}$. This is all the numbers you currently know about...

But it isn't all the numbers in Maths. Not even close.

We return back to $i=\sqrt{-1}$, we call i an imaginary number, since it is not a real number.

(Careful here, these are just names, if we had called the real numbers something else like 'hot' numbers we may have called i a 'cold' number so imaginary here doesn't mean it doesn't exist.)

Any multiple of i is also called an imaginary number, so $3i$, $-5i$, $i\sqrt{3}$ are all imaginary numbers too.

A combination of a real number and an imaginary number added together is called a complex number - complex since it has two parts, a real part and an imaginary part. For example, $4-3i$ is a complex number, with real part 4 and imaginary part $-3i$. $6i+2$ is also a complex number with real part 2 and imaginary part $6i$. We usually write the real part first and the imaginary part second so $6i+2$ will be written as $2+6i$. (Interestingly, 5 is also a complex number with real part 5 and imaginary part 0. And $4i$ is also a complex number with real part 0 and imaginary part $4i$.)

So what is the point to complex numbers? I'm glad you asked (or did I ask?) The 'answer' is there are many, many useful things you can do with them. They are fundamental to so many topics in maths, science, finance, once you reach degree level. One of the main reasons for this is because they allow you to solve equations that you couldn't solve before. Lets see some examples:

Example 1
Find the values of x that solve the equation $x^2=-4$.

Answer 1
We square root both sides $x=\sqrt{-4}$ and then use rules of surds to separate 4 and -1 $x=\sqrt{4}\sqrt{-1}$ then replace $\sqrt{-1}$ with i to get $x=\sqrt{4}i=2i$. But remember that when you square root a number there are two answers so actually $x=\pm\sqrt{4}i=\pm 2i$.

Example 2
Find the values of x that solve the equation $x^2+4x+7=0$.

Answer 2
We solve the quadratic by completing the square. If you are not sure about completing the square from GCSE revise this first.

$$x^2+4x+7=0$$
$$(x+2)^2-2^2+7=0$$
$$(x+2)^2-4+7=0$$
$$(x+2)^2+3=0$$
$$(x+2)^2=-3$$

Now we must square root both sides and proceed as we did in example 1 to simplify the negative square root:

$$(x+2)^2 = -3$$
$$x+2 = \pm\sqrt{-3}$$
$$x+2 = \pm\sqrt{3}\sqrt{-1}$$
$$x+2 = \pm\sqrt{3}i$$
$$x = -2 \pm \sqrt{3}i$$

Exercise 1A

1. Find the values of x which satisfy each equation, giving your answers in their simplest form:
   a. $x^2 = -9$
   b. $x^2 = -16$
   c. $x^2 = -2$
   d. $x^2 = -8$
   e. $x^2 + 20 = 0$
   f. $x^2 + 50 = 0$

2. By completing the square first, find the values of x which satisfy each equation, giving your answers in their simplest form:
   a. $x^2 + 2x + 5 = 0$
   b. $x^2 + 6x + 18 = 0$
   c. $x^2 - 4x + 9 = 0$
   d. $x^2 - 4x + 12 = 0$
   e. $2x^2 + 2x + 1 = 0$

Addition and Subtraction of Complex Numbers

When you add or subtract two complex numbers you treat the i like any other algebraic letter.

Example 3
Let $z = 4 + 5i$ and $w = 3 - 7i$. Find the value of:
a) $z + w$
b) $z - w$
c) $2w - 3z$

Answer 3

a)
$$\begin{aligned} & z+w \\ &= (4+5i)+(3-7i) \\ &= 4+5i+3-7i \\ &= 4+3+5i-7i \\ &= 7-2i \end{aligned}$$

b)
$$\begin{aligned} & z-w \\ &= (4+5i)-(3-7i) \\ &= 4+5i-3+7i \\ &= 4-3+5i+7i \\ &= 1+12i \end{aligned}$$

c)
$$\begin{aligned} & 2w-3z \\ &= 2(3-7i)-3(4+5i) \\ &= 6-14i-12-15i \\ &= 6-12-14i-15i \\ &= -6-29i \end{aligned}$$

Exercise 1B

3. Let $z=-5+3i$ and $w=6+2i$,. Find the value of each of the following expressed as a complex number in its simplest form:
   a. $z+w$
   b. $z-w$
   c. $-4z+5w$
   d. $3w-5z$

4. Let $z_1=a+2bi$ and $z_2=3a-bi$. Find the value of each of the following in terms of a and b in its simplest form:
   a. z_1+z_2
   b. z_1-z_2
   c. z_2-3z_1

Multiplication of Complex Numbers

When you multiply two complex numbers together you will generate i^2 terms. But since $i^2=-1$ we can substitute this into the expressions to simplify them.

Example 4
Let $z=4+5i$ and $w=3-7i$. Find the value of:
a) zw
b) z^2
c) w^3

Answer 4

a)
$$\begin{aligned}&zw\\&=(4+5i)(3-7i)\\&=4(3-7i)+5i(3-7i)\\&=12-28i+15i-35i^2\\&=12-13i-35i^2\\&=12-13i-35(-1)\\&=12-13i+35\\&=47-13i\end{aligned}$$

b)
$$\begin{aligned}&z^2\\&=(4+5i)^2\\&=(4+5i)(4+5i)\\&=4(4+5i)+5i(4+5i)\\&=16+20i+20i+25i^2\\&=16+40i+25(-1)\\&=16+40i-25\\&=-9+40i\end{aligned}$$

c)
$$\begin{aligned}&w^3\\&=(3-7i)^3\\&=(3-7i)(3-7i)(3-7i)\\&=[3(3-7i)-7i(3-7i)](3-7i)\\&=[9-21i-21i+49i^2](3-7i)\\&=[9-42i-49](3-7i)\\&=[-40-42i](3-7i)\\&=-40(3-7i)-42i(3-7i)\\&=-120+280i-126i+294i^2\\&=-120+154i-294\\&=-414+154i\end{aligned}$$

Exercise 1C

5. Let $z = -2 + 3i$ and $w = 5 + 2i$,. Find the value of each of the following expressed as a complex number in its simplest form:
   a. zw
   b. z^2
   c. $z^2 + z$
   d. $w^2 z$

6. Let $z_1 = 3a + 2i$ and $z_2 = 3 - ai$. Find the value of the following in terms of a in its simplest form:
   a. $z_1 z_2$
   b. z_1^2
   c. z_2^2

2. Sequences

A *sequence* is list of numbers, for example: 2, 5, 8, 11, 14, ...
We can express a general sequence as $u_1, u_2, u_3, u_4, u_5, \ldots$ where the u_i can be any numbers. These numbers are usually generated by a rule (often called the *n*th term), in the example sequence, 2, 5, 8, 11, 14, the *n*th term is $u_n = 3n - 1$. Which generates each term as we can show: $u_1 = 3(1) - 1 = 2$, $u_2 = 3(2) - 1 = 5$, $u_3 = 3(3) - 1 = 8$, ...

You need to be able to find a sequence given an *n*th term and also do the opposite - find the *n*th term given a sequence.

Example 1
A sequence $u_1, u_2, u_3, u_4, u_5, \ldots$ is generated by $u_n = 2n^2 + 3$ find:
a) The first 3 terms of the sequence
b) The 100th term of the sequence.

Answer 1
a) To find the first three terms we substitute n = 1, n = 2, and n = 3 in turn into the *n*th term rule: $u_1 = 2(1)^2 + 3 = 5$, $u_2 = 2(2)^2 + 3 = 11$, $u_3 = 2(3)^2 + 3 = 21$
b) To find the 100th term we substitute n = 100 into the nth term expression: $u_{100} = 2(100)^2 + 3 = 20003$.

Example 2
A sequence begins 5, 10, 20, 40, 80, and continues in this way.
a) Find an expression for the *n*th term of the sequence.
b) Find the 15th term of the sequence.
c) Find the *r*th term of the sequence in terms of r.
d) Find the (*r* + 1)-th term of the sequence in terms of r.

Answer 2
a) To start we need to identify the pattern. Here each successive number is twice the last, so the pattern is multiplication by 2. We now write out the sequence to emphasize the pattern: 5, 5x2, 5x2x2, 5x2x2x2, 5x2x2x2x2, ... each time we multiply by an extra two. If we use indices we can write this as 5, $5\text{x}2$, $5\text{x}2^2$, $5\text{x}2^3$, $5\text{x}2^4$, ... and we begin to see that the power of two increases by 1 each time. In fact we can write $5 = 5\text{x}2^0$ and $5\text{x}2 = 5\text{x}2^1$ so now the sequence is $5\text{x}2^0$, $5\text{x}2^1$, $5\text{x}2^2$, $5\text{x}2^3$, $5\text{x}2^4$, ... Writing out each term:

$$u_1 = 5 \times 2^0$$

$$u_2 = 5 \times 2^1$$

$$u_3 = 5 \times 2^2$$

$$u_4 = 5 \times 2^3$$

To find an expression for u_n we need to spot the pattern: each power of two is one less than the term number. So $u_n = 5 \times 2^{n-1}$.

b) The 15th term is given by substituting n = 15, $u_{15} = 5 \times 2^{15-1} = 5 \times 2^{14} = 81920$
c) The *r*th term is given by substituting n = *r*, $u_r = 5 \times 2^{r-1}$
d) The (*r* + 1)-th term is given by substituting n = *r* + 1, $u_{r+1} = 5 \times 2^{r+1-1} = 5 \times 2^r$.

Exercise 2A

1. A sequence $u_1, u_2, u_3, u_4, u_5, \ldots$ is generated by $u_n = 4n - 1$.
   a. Find the first three terms of the sequence.
   b. Find the 10th term of the sequence.
   c. Find the (r + 1)-th term of the sequence in terms of r.

2. A sequence $u_1, u_2, u_3, u_4, u_5, \ldots$ is generated by $u_n = 2 \times 3^n$.
   a. Find the first three terms of the sequence.
   b. Write down the value of $\frac{u_3}{u_2}$
   c. Explain why $\frac{u_{10}}{u_8} = 9$.

3. A sequence $u_1, u_2, u_3, u_4, u_5, \ldots$ is generated by $u_n = 3 - 2n$.
   a. Find the first three terms of the sequence.
   b. Find the *k*-th term of the sequence in terms of *k*.
   c. Hence, find the value of *k* such that $u_k = -31$.

4. A sequence begins 4, 9, 14, 19, 24, ... and continues in this way.
   a. Find an expression for the *n*th term of the sequence.
   b. Find the 20th term of the sequence.
   c. Find u_{n+2} in terms of n.
   d. Find the value of $u_{n+2} - u_n$.

5. A sequence begins 7, 21, 63, 189, ... and continues in this way.
   a. Find an expression for the *n*th term of the sequence.
   b. Find the value of the 10th term of the sequence.
   c. Find the value of $\frac{u_{k+1}}{u_k}$.